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Harmonic oscillator with periodic non-zero mass

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Abstract. We present two cases of a quantum harmonic oscillator with a time-dependent non-zero mass. These are considered to be more realistic than the previously studied variable masses. In particular, periodic solutions for the kinetic and potential energies are investigated by means of stability charts. The general periodic mass is discussed briefly.

1. Introduction

The study of the harmonic oscillator with a time-dependent mass arises in the description of the electric and magnetic field intensities in a Fabry-Pérot cavity, as discussed by Colegrave and Abdalla (1981a). Using time-dependent canonical transformation theory, these authors reduced the variable-mass Hamiltonian

$$H(t)=\tfrac{1}{2}p^2/M(t)+\tfrac{1}{2}M(t)\omega^2q^2 \qquad [q,p]=\mathrm{i}\hbar \tag{1.1}$$

to the standard constant-mass form. Two exactly solvable models were presented, namely an exponentially decaying mass (Colegrave and Abdalla 1981b) and an oscillating mass (Colegrave and Abdalla 1982).

A different approach, leading more directly to the solution of the corresponding Schrödinger equation:

$$\{-[\hbar^2/2M(t)]\,\partial^2/\partial q^2+\tfrac{1}{2}M(t)\omega^2q^2\}\psi=\mathrm{i}\hbar\,\partial\psi/\partial t \tag{1.2}$$

was proposed by Leach (1983). Leach first used a change of timescale for the Hamiltonian system described by (1.1), putting

$$(q,p,t)\to\left(q',p',t'\colon q'=q,\ p'=p,\ t'=\int^t M^{-1}(s)\,\mathrm{d}s\right). \tag{1.3}$$

The presence of zeros in $M(t)$ amounts to a dilatation of the timescale in (q',p',t') space. Next a point transformation in the space variables was performed, which yields the Hamiltonian of the time-independent harmonic oscillator. This generalised transformation is

$$(q',p',t')\to\left(Q,P,T\colon Q=q'/\rho,\ P=\rho p'-\dot\rho q',\ T=\int^{t'}\rho^{-2}(s)\,\mathrm{d}s\right) \tag{1.4}$$

where $\rho(t')$ is a solution of the auxiliary equation:

$$\ddot\rho+N^2\omega^2\rho=1/\rho^3 \tag{1.5}$$

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and

$$N[t'(t)] = M(t). \tag{1.6}$$

The physically relevant quantities are the expectation values for the potential and kinetic energies of the time-dependent system:

$$\langle n|V|n\rangle = \tfrac{1}{2}M\omega^2\nu^2\hbar(n+\tfrac{1}{2}) \tag{1.7a}$$

$$\langle n|T|n\rangle = \tfrac{1}{2}M^{-1}(\nu^{-2}+M^2\dot{\nu}^2)\hbar(n+\tfrac{1}{2}) \tag{1.7b}$$

where

$$\nu(t) = \rho[t'(t)]. \tag{1.8}$$

As shown by Leach (1983) this can be written as

$$\nu(t) = (A\zeta_1^2 + B\zeta_2^2 + 2C\zeta_1\zeta_2)^{1/2} \tag{1.9}$$

where ζ_1 and ζ_2 are two linearly independent solutions of the second-order differential equation

$$\ddot{\zeta} + (\omega^2 - \ddot{\eta}/\eta)\zeta = 0 \tag{1.10}$$

with

$$M(t) = \eta^2(t). \tag{1.11}$$

The parameters A, B and C depend on the initial conditions and are related by

$$AB - C^2 = \eta^{-2}W \tag{1.12}$$

where W is the Wronskian of equation (1.10).

The cases studied by Colegrave and Abdalla (1981b, 1982) are particularly simple because $\ddot{\eta}$ is a constant multiple of η and consequently equation (1.10) is trivial to solve. Several more examples of exactly solvable models were presented by Leach (1983), using functions $\eta(t)$ for which equation (1.10) has analytical solutions. However, all these examples have the disadvantage that $\eta(t)$, which is the square root of the time-dependent mass, becomes zero at a certain finite time. Although this is a case which could arise in an ideal Fabry–Pérot cavity, one would like to study masses which do not become zero. In particular, solutions for a periodically varying non-zero mass would be very interesting, not only in optics, but Colegrave and Abdalla (1982) suggest applications in other branches of physics as well.

In this paper we will show in § 2 how the choice of a zeroth-order cosine type Mathieu function for $\eta(t)$ allows us to study the present problem in terms of the known properties of the Mathieu equation. Furthermore, in § 3, we will investigate the case

$$\eta(t) = a + b\cos\alpha t \qquad a > |b| > 0 \tag{1.13}$$

proposed by Leach (1983). We then have to determine the stability of the solutions of a certain Hill equation. In § 4 we compare these two cases and present our conclusions.

2. Time-dependent mass using the Mathieu equation

It was stated by Colegrave and Abdalla (1982) that the study of a non-zero periodic mass leads to an awkward analysis and that there is no hope of an exact solution. In this section we will prove that this is not the case and that an analytically tractable model is possible indeed.

To this end we suppose that $\eta(t)$, i.e. the square root of the time-dependent mass, obeys the equation

$$\ddot{\eta}/\eta = A + B\cos\alpha t \tag{2.1}$$

or

$$\ddot{\eta} - (A + B\cos\alpha t)\eta = 0. \tag{2.2}$$

After a change of variables,

$$(t, \eta) \rightarrow (z, y\colon z = \tfrac{1}{2}\alpha t, y(z) = \eta(t)) \tag{2.3}$$

this is a Mathieu equation, which has the canonical form

$$\mathrm{d}^2 y/\mathrm{d}z^2 + (a - 2q\cos 2z)y = 0 \tag{2.4}$$

(throughout this paper we use the conventions of McLachlan (1964)) and the values of the parameters are $a = -4A/\alpha^2$, $q = 2B/\alpha^2$.

It is well known that the only periodic solutions (with period π or 2π) of Mathieu's equation (2.4) are the Mathieu functions $ce_m(z, q)$ and $se_m(z, q)$. Furthermore, ce_m has m zeros in $]0, \pi[$ and se_m has $m-1$ zeros in $]0, \pi[$ and is zero in 0 and π. Consequently, in order that $\eta(t)$ corresponds to a periodically varying non-zero mass we must choose

$$\eta(t) = ce_0(\tfrac{1}{2}\alpha t, q) \tag{2.5}$$

which is a solution of

$$\ddot{\eta} + \tfrac{1}{4}\alpha^2(a_0 - 2q\cos\alpha t) = 0 \tag{2.6}$$

where q is a parameter and a_0 is the characteristic number for ce_0, which is uniquely determined by q and can be expanded as

$$a_0 = -\tfrac{1}{2}q^2 + \tfrac{7}{128}q^4 - \tfrac{29}{2304}q^6 + \mathrm{O}(q^8). \tag{2.7}$$

This means that (q, a) is a point somewhere on the lower curve in the stability chart (figure 1).

In other words, we can arbitrarily choose B in equation (2.1), but then A is fixed:

$$A = -\alpha^2 a_0/4 \tag{2.8a}$$

where a_0 is the characteristic number corresponding to

$$q = 2B/\alpha^2. \tag{2.8b}$$

We are therefore led to investigate a one-parameter class of oscillating non-zero masses

$$M(t) = [ce_0(\tfrac{1}{2}\alpha t, q)]^2. \tag{2.9}$$

For sufficiently small values of q we have

$$M(t) = [1 - \tfrac{1}{2}q\cos\alpha t + \mathrm{O}(q^2)]^2 \tag{2.10}$$

and for larger q values one can use further terms in this expansion (McLachlan 1964). A plot of the mass variation (2.9) is shown in figure 2. Furthermore, in the limit $|q| \rightarrow +\infty$ one can prove, using the asymptotic behaviour and the normalisation of the

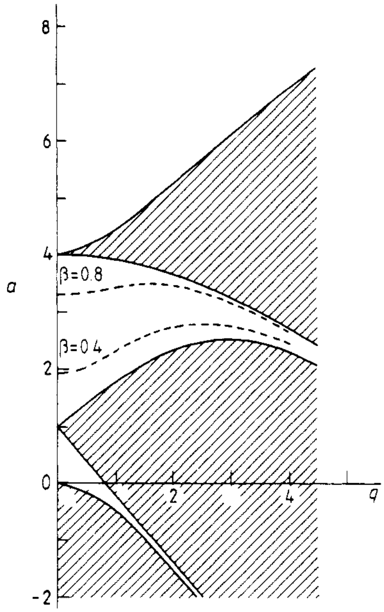

Figure 1. Stability chart for the Mathieu equation. The hatched portions correspond to regions of instability. In the second region of stability the iso-β curves for two values of β are indicated.

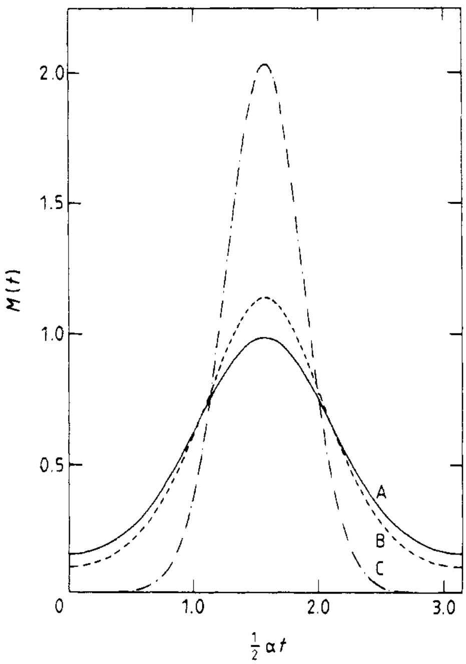

Figure 2. The mass variation $[ce_0(\frac{1}{2}t, q)]^2$ in the interval $[0, \pi]$ for three different q values: A, $q = 1$; B, $q = 2$; C, $q = 8$. The function $[ce_0(\frac{1}{2}\alpha t, q)]^2$ is periodic with period π.

Mathieu functions, that

$$M(t)=\frac{2\pi}{\alpha}\sum_{n=-\infty}^{+\infty}\delta\left(t-\frac{2n\pi}{\alpha}\right) \tag{2.11}$$

although this could hardly be considered a very realistic case.

Substitution of $\ddot{\eta}/\eta$, obtained from (2.6), in (1.10) gives the Mathieu equation

$$\ddot{\zeta}+(\omega^2+\tfrac{1}{4}\alpha^2 a_0-\tfrac{1}{2}\alpha^2 q\cos\alpha t)\zeta=0 \tag{2.12}$$

or after a change of variables

$$(t,\zeta)\rightarrow(z,y\colon z=\tfrac{1}{2}\alpha t,\ y(z)=\zeta(t)) \tag{2.13}$$

we obtain

$$\mathrm{d}^2y/\mathrm{d}z^2+(4\omega^2/\alpha^2+a_0-2q\cos 2z)y=0. \tag{2.14}$$

We have to determine two linearly independent solutions $y_1(z,q)$ and $y_2(z,q)$ in order to solve the problem. Conventionally one takes

$$y_1(0)=y_2'(0)=1 \tag{2.15a}$$

$$y_2(0)=y_1'(0)=0 \tag{2.15b}$$

and then the Wronskian is

$$y_1(z)y_2'(z)-y_2(z)y_1'(z)\equiv 1. \tag{2.16}$$

A general investigation of the solutions of equation (2.14) would be beyond the bounds of this paper and we will therefore restrict ourselves to periodic solutions.

From a physical point of view the most interesting situations are those for which the expectation values (1.7) are periodic functions of time. Obviously this will be the case when $\nu(t)$ defined in (1.9) is periodic. Because of the relation (1.12) with the Wronskian the case where only one parameter A or B is not zero can never occur. Therefore both linearly independent functions ζ_1 and ζ_2 will always contribute to $\nu(t)$, whatever the initial conditions are. Consequently $\nu(t)$ can only be periodic when ζ_1 and ζ_2 are periodic functions with the same period.

From the study of the stability of the solutions of Mathieu's equation one knows that the following cases are possible:

(i) y_1 (or y_2) has period π or 2π, then y_2 (or y_1) is not periodic;
(ii) y_1 and y_2 are both not periodic, and one is unbounded when $z\rightarrow+\infty$;
(iii) y_1 and y_2 have the same period $s\pi$, $s=3,4,\ldots$;
(iv) y_1 and y_2 are non-periodic but bounded when $|z|\rightarrow+\infty$.

The first case occurs when, for the given q, the value $a=a_0+4\omega^2/\alpha^2$ is equal to a characteristic number a_m or b_m. The second one corresponds to a (q,a) value in a region of instability and the last two to a value in a region of stability. In the stability chart (figure 1) one can view this as a 'direct transition' (i.e. with the same q value) from a point (q,a_0) to a point (q,a). The hatched region and the boundaries correspond to non-periodic solutions. In the regions of stability the solutions are characterised by the parameter $\beta(\varepsilon]0,1[)$, the general solution being

$$y(z)=A\sum_{r=-\infty}^{+\infty}c_{2r}\exp[(2r+\beta)z\mathrm{i}]+B\sum_{r=-\infty}^{+\infty}c_{2r}\exp[-(2r+\beta)z\mathrm{i}]. \tag{2.17}$$

When β is a rational fraction p/s (p and s relatively prime) the general solution has period $s\pi$. When β is irrational the solution is oscillatory, but bounded and non-periodic.

We conclude that under certain conditions the matrix elements (1.7) are periodic functions of time, with a period which is a multiple ($2s\pi/\alpha$; $s=3, 4, \ldots$) of the period of the mass (2.9).

3. Time-dependent mass using a Hill equation

Another periodically varying non-zero mass has been proposed by Leach (1983), writing

$$\eta(t)=a+b\cos\alpha t \qquad a>|b|>0. \tag{3.1}$$

Equation (1.10) is then

$$\ddot{\zeta}+[\omega^2+\alpha^2-a\alpha^2(a+b\cos\alpha t)^{-1}]\zeta=0 \tag{3.2}$$

which after the change of variables

$$(t,\zeta)\to(z,y\colon z=\tfrac{1}{2}\alpha t,\ y(z)=\zeta(t)) \tag{3.3}$$

transforms to

$$\mathrm{d}^2y/\mathrm{d}z^2+4[\beta^2-(1+\sigma\cos 2z)^{-1}]y=0 \tag{3.4}$$

where $\beta^2=(\omega^2+\alpha^2)/\alpha^2$ and $|\sigma|=|b/a|<1$ (note that β^2 here should not be confused with the β of the previous section, which is a conventional label for the fractional order solutions of Mathieu's equation).

This is of the form of Hill's equation (Magnus and Winkler 1979), which in general reads:

$$\mathrm{d}^2y/\mathrm{d}z^2+[\lambda+Q(z)]y=0 \tag{3.5a}$$

$$Q(z+\pi)=Q(z). \tag{3.5b}$$

Equation (3.4) was obtained by Leach (1983), but it was not analysed further.

Nevertheless, the stability chart of the Hill equation (3.4) can easily be obtained by numerical methods described elsewhere (Wille and Phariseau 1985) and is shown in figure 3. In brief, the stability is determined by an infinite or Hill determinant. Since the Fourier coefficients of the periodic function occurring in (3.4) are proportional, this determinant can be evaluated by means of a recurrence formula. The value of the determinant allows one to decide whether the (σ, β^2) point is situated in a region of stability or instability, or on a borderline (corresponding to a periodic solution, with period π or 2π).

Again we will focus on periodic solutions of equation (3.5). We start by noting that for $\beta^2=1$ equation (3.4) always possesses a periodic solution:

$$y_1(z)=\frac{1+\sigma\cos 2z}{1+\sigma} \tag{3.6}$$

and the second solution $y_2(z)$ is not periodic. When $\sigma=1$ equation (3.4) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+\left(4\beta^2-\frac{2}{\cos^2 z}\right)y=0 \tag{3.7}$$

which has the general solution (Kamke 1967, C2.420; see also Scarf 1958).

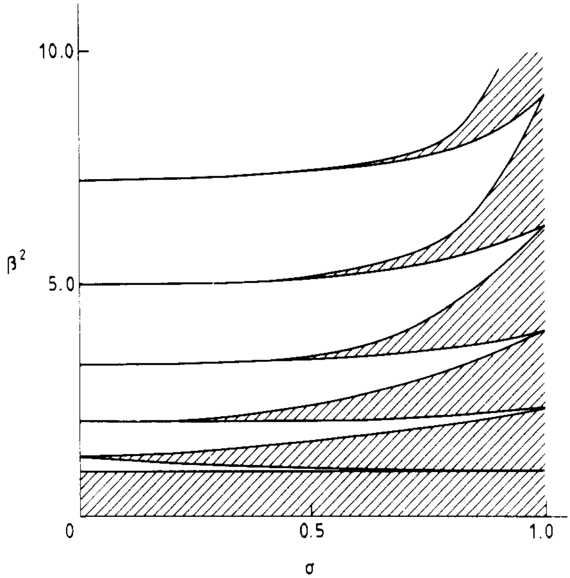

Figure 3. Stability chart for the Hill equation (3.4). The hatched portions are regions of instability.

$$y(z) = \cos^2 z\left(\frac{1}{\cos z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 [C_1 \exp(2\mathrm{i}\beta z) + C_2 \exp(-2\mathrm{i}\beta z)]. \tag{3.8}$$

These solutions are always unstable, except when

$$\beta^2 = (n+1)^2/4 \qquad n = 1, 2, \ldots \tag{3.9}$$

and then two periodic solutions exist simultaneously. In general the zeroth region of instability is the interval $]-\infty, 1[$ for all values of σ; the nth region of instability starts at $(n/2+1)^2$ $(n = 1, 2, \ldots)$ when $\sigma = 0$ and extends from $(n+1)^2/4$ to $(n+2)^2/4$ when $\sigma = 1$. Using the same conventions (2.15) as before, the Wronskian of the Hill equation (3.5) is equàl to 1. Equation (3.4) is of the form of Ince's equation (Magnus and Winkler 1979) which in general can possess two solutions of period π (or 2π) simultaneously (coexistence). However for the present equation this is not the case (except for $\sigma = 1$) and so exactly the same considerations as in § 2 can be made concerning the periodicity of the solutions y_1 and y_2. In particular, for appropriate values of σ and ω^2 the matrix elements (1.7) can have a period which is a multiple of that of the mass (3.1).

We conclude by noting that the most general periodic mass variation will have the property

$$\ddot{\eta}/\eta = -Q(t) \tag{3.10a}$$

$$Q(t+\pi) = Q(t) \tag{3.10b}$$

in appropriate time units. In other words, $\eta(t)$ is a solution of a Hill equation belonging to the eigenvalue $\lambda = 0$. Substitution in (1.10) gives

$$\ddot{\zeta} + [\omega^2 + Q(t)]\zeta = 0 \tag{3.11}$$

and so $\zeta(t)$ is a solution of the same Hill equation, but belonging to the eigenvalue $\lambda = \omega^2$. Consequently the picture of direct transitions in a stability chart is generally valid.

Conversely, any periodic solution of a Hill equation will give a possible mass variation (which, in general however, will have zeros). When the periodic solutions coexist (Magnus and Winkler 1979) the matrix elements (1.7) can oscillate in phase with the mass.

4. Conclusions

We have studied the quantum harmonic oscillator with a periodically varying mass. The expectation values of the physical observables are determined by a second-order differential equation which, for a periodically varying mass, is a Hill equation. In particular, we present two expressions for periodic masses which are always strictly positive. This is an improvement over the work of other authors (Colegrave and Abdalla 1982) where the mass returned periodically to zero.

In the first case considered the mass is the square of a zeroth-order cosine type Mathieu function. Although this is a rather artificial time dependence it has the advantage of allowing a completely analytical treatment, and as such it has the character of a model calculation. Moreover, for sufficiently small values of the parameter q, the mass can be expanded in a Fourier series and has a simple time dependence.

The second mass variation, proposed by Leach (1983), is physically more acceptable but the corresponding Hill equation must be solved numerically. In both cases we focus on periodic solutions and it is found that the expectation values can show resonant behaviour with a period which is a multiple of that of the mass.

As the main field of application of the models investigated here, we mention laser-producing or laser-driven cavities. A Fabry–Pérot cavity in contact with a reservoir of two-level atoms can be described by a Hamiltonian constructed in terms of mass parameters $M_s^{1/2}(t)$ in the various modes (Sargent *et al* 1974). In the case of single-mode operation at frequency ω this Hamiltonian assumes the familiar harmonic oscillator form (1.1) and the electric and magnetic fields are proportional to $M^{1/2}(t)$ (Colegrave and Abdalla 1981a). Evidently these field intensities are also governed by the expected number of photons in the mode considered. Pulsating fields, as they have been studied here, would have to be maintained by a pumping mechanism. Kumar and Mehta (1981) have performed calculations for pump-signal-idler systems and they obtained results in terms of Jacobian elliptic functions, which could be approximated by a mass law of the type (3.1). Since this mass law has to be treated numerically, one would like to find an analytically solvable model in order to get some insight into the properties of non-zero oscillating field intensities. As shown here, zeroth-order cosine type Mathieu functions provide such a model.

Finally, we would like to point out that the stability chart for the Hill equation (3.4) is an interesting result in its own right, which might have applications in several branches of physics. As one example we mention direct capacitance modulation in circuits involving a resistance (McLachlan 1964, 15. 30). The case of zero resistance has been discussed in the literature (Infeld 1977), but the general case has only been studied approximatively by reducing the problem to a Mathieu equation. It is not implausible that other problems of parametric resonance can be described by equations similar to (3.4).

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